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FIRST INTEGRALS OF THE EQUATIONS OF NON-LINEAR WAVE DYNAMICS†

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An algorithm for finding the first integrals of fourth-order non-linear differential equations, encountered when describing non-linear wave processes, is proposed. The use of the method is illustrated by a number of examples. The first integrals obtained are employed to construct a solution of the generalized fifth-order Korteweg–de Vries equation in travelling-wave variables, which is expressed in terms of hyperelliptic integrals. © 2005 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

Finding first integrals is, essentially, one of the main problems of mechanics and the theory of differential equations in attempts to obtain a general solution of any problem in quadratures.

Several fifth-order differential equations have been proposed to describe waves on water [1, 2]. One of these has the form

$$\eta_t + \left(\frac{5}{6} - \chi^2 - \frac{\tau}{2}\right)\left(\chi^2 - \frac{2}{3}\right)\beta^2\eta_{\xi\xi\xi\xi\xi} + \left(\frac{53}{24} - \frac{11}{4}\chi^2 - \frac{9}{8}\tau\right)\alpha\beta\eta\eta_{\xi\xi\xi} +$$

$$+ \left(\frac{1}{6} - \frac{\tau}{2}\right)\beta\eta_{\xi\xi\xi} - \left(\frac{139}{24} - 7\chi^2 + \frac{27}{8}\tau\right)\alpha\beta\eta_\xi\eta_{\xi\xi} - \frac{45}{32}\alpha^2\eta^2\eta_\xi + \frac{3}{2}\alpha\eta\eta_\xi + \eta_\xi = 0 \tag{1.1}$$

Here α and β are small parameters

$$\alpha = a/h, \quad \beta = h/l$$

a is the amplitude of the perturbation, h is the depth of the liquid, l is the wavelength, τ is the dimensionless surface tension coefficient, χ is the dimensionless distance from the surface ($0 \le \chi \le 1$), and $\eta(\xi, t)$ is the wave profile, which depends on the coordinate ξ and the time t.

For arbitrary values of the parameters α, β and τ, Eq. (1.1) does not belong to a class of exactly solvable equations, since it does not pass the Painlevé test [3]. Using an approach similar to that used previously in [3], it is possible to obtain only a certain set of particular solutions. However, for a special choice of the variables and the parameters

$$\eta = \frac{16}{3\beta'\alpha}w, \quad \xi = Lz, \quad t = -\frac{3L\beta'^2}{4}t', \quad \tau = \frac{11}{9}, \quad \chi = 0, \quad \beta = \frac{3L^2}{\beta'} \tag{1.2}$$

Eq. (1.1) can be converted to a non-linear fifth-order equation (the primes on β' and t' are omitted)

$$w_t + w_{zzzzz} - 10ww_{zzz} + \beta w_{zzz} - 20w_zw_{zz} + 30w^2w_z - 6\beta ww_z = 0 \tag{1.3}$$

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When $\beta = 0$, Eq. (1.3) is a fifth-order Korteweg–de Vries equation. The generalized modified fifth-order Korteweg–de Vries equation

$$w_t + w_{zzzzz} + \beta w_{zzz} - 40ww_z w_{zz} - 10w^2 w_{zz} - 10w_z^3 + 30w^2 w_z - 6\beta w^2 w_z = 0 \tag{1.4}$$

is connected with Eq. (1.3) by a Miura transformation [4–6].

If we seek a solution of the wave equations (1.3) and (1.4) in travelling-wave variables

$$w(z, t) = y(x), \quad x = z - c_0 t$$

then, from Eqs (1.3) and (1.4), after integration with respect to x, we arrive at the following non-linear ordinary differential equations

$$y_{xxxx} - 10yy_{xx} - 5y_x^2 + 10y^3 + \beta(y_{xx} - 3y^2) + \delta y + \mu = 0 \tag{1.5}$$

$$y_{xxxx} - 10y^2 y_{xx} - 10yy_x^2 + 6y^5 + \beta(y_{xx} - 2y^3) + \delta y + \mu = 0 \tag{1.6}$$

Here we have introduced the following notation: $\delta = -c_0$, and μ is a constant of integration.

The problem arises of finding a general solution of Eqs (1.5) and (1.6). It is well known that, to integrate N ordinary differential equations in the general case, it is necessary to know N first integrals. However, for autonomous systems it is sufficient to known N-1 first integrals, while to integrate Hamiltonian systems in quadratures, as a rule, it is sufficient to know $N/2$ first integrals, which follows from Liouville's theorem [7–9].

Finding first integrals of many non-linear differential equations (particularly of high order) often involves considerable difficulties, which is explained by the lack of a general approach to solving this problem.

In this paper we propose an algorithm for finding the first integrals of non-linear differential equations having the form of polynomials in the dependent variable and its derivatives. The algorithm consists of the following stages: (1) choosing the leading terms of the non-linear differential equation, (2) writing a polynomial with undetermined coefficients, corresponding to the first integral of the new differential equation, which contains leading terms of the initial equation, (3) solving systems of linear algebraic equations for the coefficients of the first integral of the new differential equation with leading terms of the initial differential equation, (4) writing a polynomial with undetermined coefficients corresponding to the initial equation, and (5) solving systems of algebraic equations for the coefficients of the new polynomial and representing the first integrals of the initial differential equation.

It should be noted that it will be hardly possible to realize this algorithm without using widely employed analytical calculation programmes of the MAPLE and MATHEMATICA types.

The proposed algorithm is illustrated by finding the first integrals of Eqs (1.5) and (1.6) and the following non-linear fourth-order differential equations, which are also encountered in wave dynamics

$$y_{xxxx} + 5y_x y_{xx} - 5y^2 y_{xx} - 5yy_x^2 + y^5 - \delta = 0 \tag{1.7}$$

$$y_{xxxx} - 4\frac{y_x y_{xxx}}{y} - 3\frac{y_{xx}^2}{y} + \frac{21}{2}\frac{y_x^2 y_{xx}}{y^2} - 5\delta\frac{y_{xx}}{y^2} - \frac{9}{2}\frac{y_x^4}{y^3} + 10\delta\frac{y_x^2}{y^3} + \nu y^2 - 2\delta^2\frac{1}{y^3} + \mu = 0 \tag{1.8}$$

$$y_{xxxx} - 2\frac{y_x y_{xxx}}{y} - \frac{3}{2}\frac{y_{xx}^2}{y} + 2\frac{y_x^2 y_{xx}}{y^2} - 5y^2 y_{xx} - \frac{5}{2}yy_x^2 + \frac{5}{2}y^5 - \beta y^3 + \mu y = 0 \tag{1.9}$$

Equation (1.7) is the stationary case of the Sawada–Kotera equation, used to describe a number of wave processes [10]. If, on the right-hand side of Eq. (1.7), we add the expression αxy, where α is a parameter, and y and x are dependent and independent variables, then Eq. (1.7) defines a new non-classical special function, expressed in terms of the solution of a fourth-order non-linear differential equation [11–13]. Equation (1.5) is widely known when $\beta = 0$. It is noteworthy that, when adding the expression αx to the right-hand side (α is a parameter and x is the variable), we also define new special functions. All the differential equations indicated above are special cases of a differential equation which arises in the Hénon and Heiles model for describing the behaviour of a star in the middle of the galactic field [14–16]. Equations (1.8) and (1.9), obtained in [17–19], arise as special cases of higher analogues of Painlevé equations, defining new special functions.

In order to construct general solutions of Eqs (1.5)–(1.9) in quadratures, it is necessary to obtain at least two first integrals of these equations.

2. AN ALGORITHM FOR FINDING FIRST INTEGRALS

In fact, all the differential equations mentioned above have one fairly obvious first integral, which can be found by multiplying the differential equation by the derivative y_x and dividing by a certain power of y. However, one first integral cannot be so simply obtained. Hence, we will use a single approach to find the first integrals of all the differential equations mentioned above.

We will consider the determination of the first integral of a differential equation. For convenience we will introduce the following notation

$$y = y_0, \quad y_x = y_1, \quad y_{xx} = y_2, \quad y_{xxx} = y_3, \quad y_{xxxx} = y_4$$

Suppose it is required to obtain the first integral of the following non-linear differential equation

$$y_4 = E(y_0, y_1, y_2, y_3) \tag{2.1}$$

Suppose also that there is a first integral of Eq. (2.1) in the form

$$P(y_0, y_1, y_2, y_3) = C_1 \tag{2.2}$$

where C_1 is an arbitrary constant.

According to the definition of the first integral for an autonomous differential equation, the first integral (2.2) satisfies the following partial differential equation

$$\sum_{n=0}^{3} y_{n+1}\frac{\partial P}{\partial y_n} = Q(y_0, y_1, y_2, y_3, y_4)(y_4 - E(y_0, y_1, y_2, y_3)) \tag{2.3}$$

where $Q(y_0, y_1, y_2, y_3, y_4)$ is a certain expression, which may depend on the variable y and its derivatives. Bearing in mind that y_4 satisfies Eq. (2.1), we obtain from Eq. (2.3)

$$\sum_{n=0}^{2} y_{n+1}\frac{\partial P}{\partial y_n} + E(y_0, y_1, y_2, y_3)\frac{\partial P}{\partial y_3} = 0 \tag{2.4}$$

This is a fundamental equation from which one can find the first integrals of the non-linear differential equations mentioned above.

To carry out the first stage of the proposed algorithm, we substitute the following expressions into the differential equation being investigated

$$\begin{aligned} y_0 &= B_0/x^p, \quad y_1 = -pB_0/x^{p+1}, \quad y_2 = p(p+1)B_0/x^{p+2} \\ y_3 &= -p(p+1)(p+2)B_0/x^{p+3}, \quad y_4 = p(p+1)(p+2)(p+3)B_0/x^{p+4} \end{aligned} \tag{2.5}$$

It should be noted that formulae (2.5) are identical with the corresponding formulae of the Painlevé–Kovalevskaya algorithm when analysing non-linear differential equations on the Painlevé property. Comparing the terms of the differential equation and choosing the least degree in the expressions obtained, we obtain a value of the exponent p (for an exactly solved differential equation this value, as a rule, is always equal to an integer, usually unity or two). Carrying out this procedure, we obtain that $p = 2$ for Eq. (1.5) and $p = 1$ for Eqs (1.6)–(1.9). For example, all terms but the last are leading terms of Eq. (1.7).

The differential equation made up of the leading terms of Eq. (1.7), takes the form

$$y_{xxxx} + 5y_x y_{xx} - 5y^2 y_{xx} - 5yy_x^2 + y^5 = 0 \tag{2.6}$$

Substituting expression (2.5) with $p = 1$ into Eq. (2.6), we obtain that all the terms of this differential equation have degree x^{-5}. It is natural to expect that the first integrals of Eq. (2.6) will have the form of polynomials, also consisting of terms of one degree when (2.5) is substituted, and they will be of less degree than the terms of Eq. (2.6). Hence, the next stage in obtaining the first integrals of Eq. (1.7) is to construct a polynomial with undetermined coefficients, all the terms of which have the same, but lower degree, than Eq. (2.6). This form of polynomial can be constructed using the following recurrence formula

$$
\begin{aligned}
&P_{k+4} = y_3 P_k + y_2 P_{k+1} + y_1 P_{k+2} + y_0 P_{k+3}, \quad k = 0, \ldots, n \\
&P_0 = 1, \quad P_1 = y_0, \quad P_2 = y_1 + y_0^2, \quad P_3 = y_0^3 + y_0 y_1 + y_2
\end{aligned}
\tag{2.7}
$$

As a result of calculations using formula (2.7) we obtain a polynomial, each term of which, when expressions (2.5) is substituted into it, has the same degree. To construct the simplest first integrals of the differential equations mentioned above, one must choose a polynomial for which $n \geq 6$.

In this paper, when obtaining the first integrals of all the differential equations, we use a polynomial in which $n = 12$. In the polynomial obtained using formulae (2.7), for each of the terms we introduce undetermined coefficients, the finding of which leads to the construction of the first integral of the corresponding differential equation. Below we use a polynomial with undetermined coefficients in the form

$$
\begin{aligned}
P_{12} = {}& a_0 y_3^3 + (a_1 y_0 y_2 + a_2 y_1^2 + a_3 y_0^2 y_1 + a_4 y_0^4) y_3^2 + (a_5 y_1 + a_6 y_0^2) y_2^2 y_3 + \\
&+ (a_7 y_0 y_1^2 + a_8 y_0^3 y_1 + a_9 y_0^5) y_2 y_3 + (a_{10} y_1^4 + a_{11} y_0^2 y_1^3 + a_{12} y_0^4 y_1^2 + a_{13} y_0^6 y_1 + a_{14} y_0^8) y_3 + \\
&+ b_0 y_2^4 + (b_1 y_0 y_1 + b_2 y_0^3) y_2^3 + (b_3 y_1^3 + b_4 y_0^2 y_1^2 + b_5 y_0^4 y_1 + b_6 y_0^6) y_2^2 + \\
&+ (b_7 y_0 y_1^4 + b_8 y_0^3 y_1^3 + b_9 y_0^5 y_1^2 + b_{10} y_0^7 y_1 + b_{11} y_0^9) y_2 + \\
&+ c_0 y_1^6 + (c_1 y_1^4 + c_2 y_0^2 y_1^3 + c_3 y_0^4 y_1^2 + c_4 y_0^6 y_1 + c_5 y_0^8) y_0^2 y_1 + c_6 y_0^{12}
\end{aligned}
\tag{2.8}
$$

Polynomial (2.8) with undetermined coefficients $a_0, \ldots, a_{14}, b_0, \ldots, b_{11}, c_0, \ldots, c_6$ was used to construct the first integrals of Eq. (2.6) in the form

$$
P = P_{12}/y_0^{12-j} = C_1, \quad j = 6, \ldots, 12
\tag{2.9}
$$

Substituting expression (2.9) into Eq. (2.6) with $j = 6$, and determining the coefficients of polynomial (2.8) by solving linear algebraic equations, we obtain the simplest first integral of Eq. (2.6).

$$
P(y_0, y_1, y_2, y_3) \equiv y_1 y_3 - \frac{1}{2} y_2^2 + \frac{5}{3} y_1^3 - \frac{5}{2} y_0^2 y_1^2 + \frac{1}{6} y_0^6 = C_1
\tag{2.10}
$$

It can be seen that the order of the power of each term in (2.10), after substituting expressions (2.5) into it, is equal to six. It follows from Eq. (2.10) that the simplest first integral of Eq. (1.7) takes the form

$$
P(y, y_x, y_{xx}, y_{xxx}) - \delta y = K_1
\tag{2.11}
$$

Here K_1 and henceforth K_2, K_3 and K_4 are arbitrary constants.

Solving the algebraic equations for the coefficients $a_0, \ldots, a_{14}, b_0, \ldots, b_{11}, c_0, \ldots, c_6$ of polynomial (2.8), we find that all these coefficients are equal to zero when $j = 7, 8, 9, 10, 11$. However, when $j = 12$ we obtain a polynomial which can be represented in the form of one more independent first integral

$$
\begin{aligned}
&y_3^3 + \left(\frac{3}{2} y_0^4 - 9 y_0^2 y_1\right) y_3^2 + F_1(y_0, y_1, y_2) y_3 - \frac{15}{8} y_2^4 + 2 y_0^3 y_2^3 + \\
&+ F_2(y_0, y_1) y_2^2 + F_3(y_0, y_1) y_2 + F_4(y_0, y_1) = C_2
\end{aligned}
\tag{2.12}
$$

where

$$
F_1(y_0, y_1, y_2) = \left(\frac{15}{2} y_1 - 3 y_0^0\right) y_2^2 + 3(y_1^2 - 2 y_0^2 y_1 + y_0^4) y_0 y_2 - 7 y_1^4 - \frac{9}{2} y_0^2 y_1^3 + 30 y_0^4 y_1^2 - \frac{17}{2} y_0^6 y_1
$$

$$F_2(y_0, y_1) = \frac{25}{2}y_1^3 + 15y_0^4y_1 - \frac{117}{4}y_0^2y_1^2 - \frac{13}{4}y_0^6$$

$$F_3(y_0, y_1) = 9y_0y_1^4 - 30y_0^3y_1^3 + 36y_0^5y_1^2 - 18y_0^7y_1 + 3y_0^9$$

$$F_4(y_0, y_1) = -\frac{22}{3}y_1^6 + \frac{35}{2}y_0^2y_1^5 + \frac{45}{8}y_0^4y_1^4 - \frac{157}{6}y_0^6y_1^3 + \frac{19}{4}y_0^8y_1^2 + 3y_0^{10}y_1 - \frac{17}{24}y_0^{12}$$

In Eq. (2.12), C_2 is an arbitrary constant. It can be seen from expression (2.12) that each term of this differential equation has an order of singularity equal to twelve. To find the first integral of the initial equation we must use a new polynomial with undetermined coefficients in the form $\delta P_7 + \delta^2 P_2$. As a result we obtain one more first integral of Eq. (1.7)

$$y_{xxx}^3 + \left(\frac{3}{2}y^4 - 9y^2y_x\right)y_{xxx}^2 + \tilde{F}_1(y, y_x, y_{xx})y_{xxx} - \frac{15}{8}y_{xx}^4 + 2y^3y_{xx}^3 +$$
$$+ \tilde{F}_2(y, y_x)y_{xx}^2 + \tilde{F}_3(y, y_x)y_{xx} + \tilde{F}_4(y, y_x) = K_2 \tag{2.13}$$

where

$$\tilde{F}_1(y, y_x, y_{xx}) = F_1(y, y_x, y_{xx}) - 3\delta y_{xx} + 9\delta yy_x, \quad \tilde{F}_2(y, y_x) = F_2(y, y_x) - \frac{9}{2}\delta y$$

$$\tilde{F}_3(y, y_x) = F_3(y, y_x) - 9\delta y_x^2 + 18\delta y^2y_x - 3\delta y^4$$

$$\tilde{F}_4(y, y_x) = F_4(y, y_x) + 7\delta yy_x^3 - \frac{27}{2}\delta y^3y_x^2 - (6\delta y^5 - 3\delta^2)y_x - \frac{9}{2}\delta^2y^2$$

Substituting y_{xxx} from Eq. (2.11) into Eq. (2.13), we obtain a second-order differential equation of the sixth degree, which is also the first integral of Eq. (1.7). The two first integrals obtained are sufficient to express the general solution of Eq. (1.7) in terms of hyperelliptic integrals, since the initial differential equation can be written in the form of a Hamiltonian system.

3. FIRST INTEGRALS OF EQS (1.5), (1.6), (1.8) AND (1.9)

The procedure for finding the first integrals of Eqs (1.5), (1.6), (1.8) and (1.9) is largely similar to that described in the previous section for Eq. (1.7). However, substitution of expressions (2.5) into Eq. (1.5) shows that the degree of the singular equation is equal to two. Consequently, the corresponding equation, consisting of the leading terms of Eq. (1.5), takes the form

$$y_{xxxx} - 10yy_{xx} - 5y_x^2 + 10y^3 = 0 \tag{3.1}$$

The polynomial with undetermined coefficients for finding the first integrals of Eq. (3.1) takes the form

$$P_{12} = A_0y_3^2y_0 + A_1y_3y_2y_1 + A_2y_3y_1y_0^2 + A_3y_2^3 + A_4y_2^2y_0^2 +$$
$$+ A_5y_2y_1^2y_0 + A_6y_2y_0^4 + A_7y_1^4 + A_8y_1^2y_0^3 + A_9y_0^6 \tag{3.2}$$

Formula (2.9) is then used, and we obtain the first integrals of Eq. (3.1) with orders of the degrees of the leading terms of x^{-8} and x^{-10}. Bearing in mind the additional polynomials, the first integral can be written in the form

$$y_xy_{xxx} - \frac{1}{2}y_{xx}^2 - 5yy_x^2 + \frac{5}{2}y^4 + \frac{1}{2}\beta(y_x^2 - 2y^3) + \frac{1}{2}\delta y^2 + \mu y = K_1 \tag{3.3}$$

One more first integral of Eq. (1.5) can be represented in the form

$$y_{xxx}^2 - 12yy_xy_{xxx} - 4yy_{xx}^2 + 2y_x^2y_{xx} + 20y^3y_{xx} + 30y^2y_x^2 - 24y^5 +$$

$$+\beta(y_{xx} - 3y^2)^2 + \delta(2yy_{xx} - y_x^2 - 4y^3) + 2\mu(y_{xx} - 3y^2) = K_2 \quad (3.4)$$

These two first integrals are sufficient to express the general solution of Eq. (1.5) in terms of hyperelliptic integrals.

Substituting formulae (2.5) into Eq. (1.6), we obtain the order of the degree of the singular equation, equal to unity. The leading terms of the differential equation have degree x^{-5}. To find the first integrals we also used polynomial (2.8) of the twelfth degree. The first integrals obtained take the form

$$y_xy_{xxx} - \frac{1}{2}y_{xx}^2 - 5y^2y_x^2 + y^6 + \frac{1}{2}\beta(y_x^2 - y^4) + \frac{1}{2}\delta y^2 + \mu y = K_1 \quad (3.5)$$

$$y_{xxx}^2 - 12y^2y_xy_{xxx} - 4y^2y_{xx}^2 + 4yy_x^2y_{xx} + 12y^5y_{xx} - y_x^4 + 30y^4y_x^2 - 9y^8 +$$

$$+\beta(y_{xx} - 2y^3)^2 + \delta(2yy_{xx} - y_x^2 - 3y^4) + 2\mu(y_{xx} - 2y^3) = K_2 \quad (3.6)$$

We used the algorithm described above to obtain the first integrals of Eqs (1.8) and (1.9). The first integral of Eq. (1.8) takes the form

$$\frac{y_xy_{xxx}}{y^2} - \frac{1}{2}\frac{y_{xx}^2}{y^2} - 2\frac{y_x^2y_{xx}}{y^3} + \frac{9}{8}\frac{y_x^4}{y^4} - \frac{5}{2}\delta\frac{y_x^2}{y^4} + \frac{1}{2}\delta^2\frac{1}{y^4} - \mu\frac{1}{y} + \nu y = K_1 \quad (3.7)$$

When finding the first integral (3.7) we first obtained the first integral of the equation consisting of the leading terms, and then part of the integral which takes into account the terms with the parameters.

One more first integral of Eq. (1.8) has the form

$$\frac{y_{xxx}^2}{y^2} - 6\frac{y_xy_{xx}y_{xxx}}{y^3} + 3\frac{y_x^2y_{xxx}}{y^4} + 9\frac{y_x^2y_{xx}^2}{y^4} - 9\frac{y_x^4y_{xx}}{y^5} + \frac{9}{4}\frac{y_x^6}{y^6} -$$

$$-2\delta\left(3\frac{y_xy_{xxx}}{y^4} + \frac{y_{xx}^2}{y^4} - 10\frac{y_x^2y_{xx}}{y^5} + \frac{19}{4}\frac{y_x^4}{y^6}\right) - \delta^2\left(4\frac{y_{xx}}{y^5} - 11\frac{y_x^2}{y^6}\right) - 2\delta^3\frac{1}{y^6} +$$

$$+2\nu y_{xx} - 3\nu\frac{y_x^2}{y} + 2\mu\frac{y_{xx}}{y^2} - \mu\frac{y_x^2}{y^3} + 6\delta\nu\frac{1}{y} + 2\delta\mu\frac{1}{y^3} = K_2 \quad (3.8)$$

The first integral (3.8) was obtained taking into account the singular part of Eq. (1.8), which was initially obtained for a differential equation composed of the leading terms (the differential equation in this case can be obtained from Eq. (1.8) assuming $\delta = \mu = \nu = 0$). The first integral obtained is identical with Eq. (3.8) when $\delta = \mu = \nu = 0$. We further add to the expression obtained the additional polynomials of lower degree, which take into account the parameters δ, μ and ν.

The first integral of (1.9), obtained using the algorithm proposed above, takes the form

$$\frac{y_xy_{xxx}}{y} - \frac{1}{2}\frac{y_{xx}^2}{y} - \frac{y_x^2y_{xx}}{y^2} - \frac{5}{2}yy_x^2 + \frac{1}{2}y^5 - \frac{1}{3}\beta y^3 + \mu y = K_1 \quad (3.9)$$

When finding the first integral (3.9) we also bore in mind the dimensionality of the parameters β and μ. One other first integral of Eq. (1.9) has the form

$$\frac{y_{xxx}^2}{y^2} - 2\frac{y_xy_{xx}y_{xxx}}{y^3} - 8y_xy_{xxx} - \frac{1}{3}\frac{y_{xx}^3}{y^3} + \frac{y_x^2y_{xx}^2}{y^4} - y_{xx}^2 + 11\frac{y_x^2y_{xx}}{y} + 5y^3y_{xx} +$$

$$+10y^2y_x^2 - \frac{10}{3}y^6 + 2\mu\frac{y_{xx}}{y} - 4\mu y^2 - 2\beta yy_{xx} + 2\beta y_x^2 + 2\beta y^4 = K_2 \quad (3.10)$$

The integrals presented above can be used to construct general solutions of Eqs (1.5)–(1.9).

4. THE GENERAL SOLUTION OF EQ. (1.5)

To illustrate the algorithm we will consider the general solution of Eq. (1.5). The solution of Eq. (1.5) with $\beta = 0$ was obtained for the first time by Drach [20], and the general solution of this differential equation was then rediscovered by others [21, 22].

We will obtain the general solution of Eq. (1.5) when $\beta \neq 0$. We will use the variables

$$H = y_{xx} - 3y^2 - \frac{1}{2}\delta, \quad I = yy_{xx} - \frac{1}{2}y_x^2 - 3y^3 - \frac{1}{2}\mu$$

The first integrals (3.3) and (3.4) can then be represented in the form

$$y_x H_x - \left(y^2 + \frac{1}{2}\delta - \beta y\right)H - (\beta + 2y)J - \frac{1}{2}H^2 + \frac{1}{2}\beta\delta y = K_1 \tag{4.1}$$

$$H_x^2 + \beta H^2 - 4HJ + \beta\delta H = K_2 \tag{4.2}$$

Suppose $P(t)$ is the hyperelliptic curve of the second kind

$$P(t) = t^5 + m_0 t^4 + m_1 t^3 + m_2 t^2 + m_3 t + m_4$$

Here m_0, m_1, m_2, m_3, m_4 are undetermined coefficients. Substituting the expression for $J(x)$ from (4.2) into (4.1) and assuming

$$y = \frac{1}{2}(u(x) + v(x) - \beta), \quad H(x) = \frac{1}{2}u(x)v(x)$$

with the condition that

$$(u - v)u_x = \sqrt{P(u)} \quad (u - v)v_x = -\sqrt{P(v)} \tag{4.3}$$

we obtain from the first integrals (4.1) and (4.2)

$$P(t) = t^5 - 3\beta t^4 + (3\beta^2 + 2\delta)t^2 - 4\mu t^2 + 2(\beta^2\delta + 4K_1)t + 4K_2 \tag{4.4}$$

From Eqs (4.3) we obtain the system [23, 24]

$$I_0(u(x)) + I_0(v(x)) = K_3, \quad I_1(u(x)) + I_1(v(x)) = x + K_4 \tag{4.5}$$

$$I_0(w) = \int_\infty^w \frac{dt}{\sqrt{P(t)}}, \quad I_1(w) = \int_\infty^w \frac{tdt}{\sqrt{P(t)}}$$

which is identical with the Jacobi integrals that occur in transformation theory [23]. The solution of system (4.5) is similar to the solution obtained by Kovalevskaya to describe the motion of a rigid body about a fixed point [25, 26]. This solution is a meromorphic function, is expressed explicitly in terms of the Riemann theta-function and is constructed on curve (4.4). For certain relations between the parameters of Eq. (1.5) we can write special cases of the solution of this equation using hyperelliptic integrals (4.5).

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